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On $(p, q, \mu, \nu, \phi_1, \phi_2)$-generalized oscillator algebra and related bibasic hypergeometric functions

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Abstract

This paper provides the generalization of the work by Floreanini *et al* (1993 *J. Phys. A: Math. Gen.* **26** 611–4) who generated bibasic hypergeometric functions from (p, q)-oscillators. We consider a six-parameter deformed oscillator algebra realized from the (p, q)-deformed boson oscillators. We build the corresponding Fock space representation in an infinite-dimensional subspace of the Hilbert space of a harmonic oscillator. We also discuss the properties of a discrete spectrum of the Hamiltonian of the deformed harmonic oscillator corresponding to this system. We then define a realization of the deformed algebra in terms of a generalized derivative and investigate the relation between this representation and generalized bibasic Laguerre functions and polynomials.

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1. Introduction

The deformation of quantum algebras and its study continue to be at the core of many investigations in mathematics and physics. The interest to quantum deformations, in particular to the quantum (p, q)-deformations of Lie algebras, is connected with the possible applications in the quantum field theories (conformal, topological field theories, etc) and quantum groups. For a nice description of two-parameter quantum groups and their representations; see, for example, [1, 2] and references therein.

At the study of the quantum groups and algebras, it became evident their connection with the noncommutative geometry and other branches of mathematics. One of the fruitful studies, besides their representations, is their relations with the theory of special functions. For instance, it was shown that q-oscillator algebra [10] (resp. (p, q)-oscillator algebra [1]) provides an algebraic interpretation of various q-special functions [9] (resp. (p, q)-special functions [3]). The algebraic interpretation of many q-special functions (resp. (p, q)-special functions) has been shown to proceed in analogy with the Lie theory treatment of their classical counterparts [12]: one considers deformed exponentials of the generators of a deformed algebra and observes that their matrix elements in representation spaces are expressible in terms of deformed special functions; one then uses models to derive properties of these functions through symmetry techniques. This approach has proved to be very fruitful. It is the one we adopt here, following [3] where the authors have examined the relation between the representation theory of a two-parameter deformation of the oscillator algebra and certain bibasic Laguerre functions and polynomials.

This paper is organized as follows. In section 2, we briefly review the (p, q)-oscillator, provide with its generalization involving six parameters and build the corresponding Fock representation. In section 3, we discuss the properties of discrete spectrum of the Hamiltonian of the deformed oscillator corresponding to this oscillator-like system. In section 4, we generate a representation of the new algebra in terms of derivatives. We then compute the matrix elements of deformed exponentials related to the annihilation and creation operators, and deduce the relation between this representation and generalized bibasic Laguerre functions and polynomials.

2. Generalized (p, q)-oscillator algebra and its Fock space representation

The (p, q)-oscillator algebra is generated by three elements A, $A^\dagger$ and N obeying relation [1]

$$AA^\dagger - p^{-1}A^\dagger A = q^N \qquad AA^\dagger - qA^\dagger A = p^{-N} \tag{1a}$$

$$[N, A] = -A \qquad [N, A^\dagger] = A^\dagger, \tag{1b}$$

where A, $A^\dagger$ are identified, respectively, as the deformed annihilation and creation operators of a (p, q)-oscillator, $N = a^\dagger a$ is the excitation number operator of a conventional (nondeformed) boson oscillator ($[a, a^\dagger] = 1$) and p, q are independent deformation parameters. In general, these two parameters may be real or a phase factor. In the following, we take throughout p and q to be real and positive. It should be noted that, in the limit $p \to 1$, algebras (1a) and (1b) reduce to the defining relations of the maths-type q oscillator and for $p = q$, one recovers the physics-type q oscillator.

The relation of A and $A^\dagger$ in terms of the conventional boson operators a and $a^\dagger$ is given by

$$A = af(N; p, q) = a\sqrt{\frac{[N]_{p,q}}{N}} \qquad A^\dagger = f(N; p, q)a^\dagger = \sqrt{\frac{[N]_{p,q}}{N}}a^\dagger, \tag{2}$$

where the symbol $[N]_{p,q}$ is expressed by

$$[N]_{p,q} = \frac{q^N - p^{-N}}{q - p^{-1}}. \tag{3}$$

As can be seen, the deformation function $f(N; p, q)$ has no zeros at positive integer eigenvalues of N (including zero). So, the deformed annihilation operator A has a single vacuum state, i.e., $|0\rangle$, like the operator a. On the other hand, for those deformed operators $A's$ for which the function $f(N; p, q)$ has zeros at positive integer eigenvalues of N, there is a set of vacuum states. In this case, if we assume that the operator A annihilates a set of number states $|n_i\rangle$, $i = 1, 2, \ldots, k$, then, we can construct a sector S_i by repeatedly applying $A^\dagger$ on the number state $|n_i\rangle$. Thus, we have k sectors corresponding to the states that are annihilated by A. A given sector may turn out to be either finite or infinite dimensional. In particular, the infinite-dimensional sectors are of special interest. One of the reasons is that, in each infinite-dimensional sector, it is possible to construct an operator, say $G^\dagger$, which is the canonical conjugate of A, i.e., $[A, G^\dagger] = \mathbb{I}$. However, in the finite-dimensional sectors the construction does not apply [4]. This plays an important role in the construction of coherent states associated with the deformed algebra.

Let us now deal with the generalization of the relations (1a) and (1b) as follows:

$$\frac{q^\nu}{p^\mu} A A^\dagger - q A^\dagger A = \left(\frac{p^{\mu-1}}{q^\nu}\right)^N \phi_1(p, q) \tag{4a}$$

$$\frac{q^\nu}{p^\mu} A A^\dagger - p^{-1} A^\dagger A = \left(\frac{p^{\mu}}{q^{\nu-1}}\right)^N \phi_2(p, q) \tag{4b}$$

$$[N, A] = -A \qquad [N, A^\dagger] = A^\dagger, \tag{4c}$$

where ϕ_1 and ϕ_2 are two non-singular and real-valued positive functions of deformation parameters satisfying the following inequalities:

$$\phi_1(p, q) > \phi_2(p, q) \qquad \text{for} \quad Q = pq > 1 \tag{5}$$

$$\phi_1(p, q) < \phi_2(p, q) \qquad \text{for} \quad Q = pq < 1. \tag{6}$$

We also assume that there exists an integer k_0 such that $\phi_1(p, q) = (pq)^{k_0}\phi_2(p, q)$. The parameters μ and ν are real numbers. It is worth noticing that if $\nu = \mu = 0$ and $\phi_1(p, q) = \phi_2(p, q) \equiv 1$, one recovers the relations (1a).

The harmonic oscillator realization of the generalized deformed oscillator (4a) and (4b) in their simplest form

$$A = a f(N; p, q, \mu, \nu) \qquad A^\dagger = f(N; p, q, \mu, \nu) a^\dagger \tag{7}$$

looks as

$$A = a \sqrt{\left(\frac{p^\mu}{q^\nu}\right)^N \frac{q^N \phi_2(p, q) - p^{-N}\phi_1(p, q)}{N(q - p^{-1})}} \tag{8}$$

$$A^\dagger = \sqrt{\left(\frac{p^\mu}{q^\nu}\right)^N \frac{q^N \phi_2(p, q) - p^{-N}\phi_1(p, q)}{N(q - p^{-1})}}\, a^\dagger. \tag{9}$$

There are two vacua for the deformed operator A, namely, the ground state $|0\rangle$ and the number state $|k_0\rangle$ such that

$$k_0 = \frac{1}{\ln(pq)} \ln\left(\frac{\phi_1(p, q)}{\phi_2(p, q)}\right). \tag{10}$$

One can readily show that conditions (5) and (6) guarantee that the integer number k_0 is non-negative. In this manner, we have two sectors S_0 and S_{k_0} which are constructed by repeatedly applying $A^\dagger$ on $|0\rangle$ and $|k_0\rangle$, respectively. The sector S_0 is of finite dimension spanned by the states $|0\rangle, |1\rangle, |2\rangle, \ldots, |k_0 - 1\rangle$; on the other hand, the infinite-dimensional sector S_{k_0} is spanned by the states $|k_0\rangle, |k_0 + 1\rangle, \ldots$ and gives a bosonic representation of algebras (4a), (4b) and (4c) without the first finite Fock states.

Let us construct the Fock representation of the generalized oscillator (4a), (4b) and (4c) in the sector S_{k_0}. We take $\{|n\rangle \equiv |k_0 + m\rangle; m = 0, 1, 2, \ldots\}$ as the complete orthonormal set of number states. One finds

$$A|n\rangle = \sqrt{\left(\frac{p^{\mu}}{q^{\nu}}\right)^{n} p^{-k_0}\phi_1(p,q)\{n-k_0\}_{p,q}}|n-1\rangle$$
$$= \sqrt{\left(\frac{p^{\mu}}{q^{\nu}}\right)^{n} q^{k_0}\phi_2(p,q)\{n-k_0\}_{p,q}}|n-1\rangle. \tag{11}$$

$$A^{\dagger}|n\rangle = \sqrt{\left(\frac{p^{\mu}}{q^{\nu}}\right)^{n+1} p^{-k_0}\phi_1(p,q)\{n-k_0+1\}_{p,q}}|n+1\rangle$$
$$= \sqrt{\left(\frac{p^{\mu}}{q^{\nu}}\right)^{n+1} q^{k_0}\phi_2(p,q)\{n-k_0+1\}_{p,q}}|n+1\rangle \tag{12}$$

$$|n\rangle \equiv |k_0+m\rangle$$
$$= \left(\frac{p^{\mu}}{q^{\nu}}\right)^{-m(m+2k_0+1)/4} \frac{(p^{-k_0}\phi_1(p,q))^{-m/2}}{\sqrt{\{m\}_{p,q}!}}(A^{\dagger})^m|k_0\rangle$$
$$= \left(\frac{p^{\mu}}{q^{\nu}}\right)^{-m(m+2k_0+1)/4} \frac{(q^{k_0}\phi_2(p,q))^{-m/2}}{\sqrt{\{m\}_{p,q}!}}(A^{\dagger})^m|k_0\rangle$$
$$= \sqrt{\frac{k_0!}{(k_0+m)!}}(a^{\dagger})^m|k_0\rangle, \tag{13}$$

where $\{m\}_{p,q}! = \{m\}_{p,q}\{m-1\}_{p,q}\ldots\{1\}_{p,q}$, $\{0\}_{p,q}! \equiv 1$ and $\{m\}_{p,q} = (q^m - p^{-m})/(q-p^{-1})$.

As required, the radicands in (11) and (12) are always positive, since does the number $\{m\}_{p,q} = (q^m - p^{-m})/(q-p^{-1})$, $m \in \mathbb{N}$. Therefore, the operators A and $A^{\dagger}$, expressed by (8) and (9), acting on the space $\{|n\rangle\}$, are well defined.

The corresponding number operator $N^{\mu,\nu}_{p,q}$ reads

$$N^{\mu,\nu}_{p,q} = \sum_{r=1}^{m} \frac{(p^{-1}-q)^r}{p^{-r}-q^r}\left(\frac{p^{\mu-1}}{q^{\nu}}\right)^{r(r-1)/2-Nr} p^{-(k_0+1)r}(q^{k_0}\phi_2(p,q))^{-r}(A^{\dagger})^r A^r \tag{14}$$

such that

$$N^{\mu,\nu}_{p,q}|k_0+m\rangle = m|k_0+m\rangle. \tag{15}$$

For $k_0 = 0$, $(\phi_1 = \phi_2 \equiv 1)$ and $\mu = \nu = 0$, relations (11)–(15) reduce to the corresponding relations for the (p,q)-deformed oscillator defined in [1].

3. Spectrum of Hamiltonian of $(p, q, \mu, \nu, \phi_1, \phi_2)$-deformed oscillator

The Hamiltonian of the $(p, q, \mu, \nu, \phi_1, \phi_2)$-deformed oscillator (4*a*), (4*b*) and (4*c*) can be defined in the same way as in the case of the standard q-deformed oscillator. From relations (4*a*) and (4*b*) we have

$$A^{\dagger}A|n\rangle = \left(\frac{p^{\mu}}{q^{\nu}}\right)^{n} \frac{q^n\phi_2(p,q) - p^{-n}\phi_1(p,q)}{q-p^{-1}}|n\rangle \tag{16a}$$

$$AA^{\dagger}|n\rangle = \left(\frac{p^{\mu}}{q^{\nu}}\right)^{n+1} \frac{q^{n+1}\phi_2(p,q) - p^{-(n+1)}\phi_1(p,q)}{q-p^{-1}}|n\rangle. \tag{16b}$$

The Hamiltonian

$$H = A^\dagger A + AA^\dagger \tag{17}$$

of the $(p, q, \mu, \nu, \phi_1, \phi_2)$-deformed oscillator has the diagonal form in the basis $\{|n\rangle \equiv |m + k_0, m = 0, 1, \ldots\}$:

$$H|n\rangle = \epsilon_n|n\rangle, \tag{18}$$

where

$$\epsilon_n = \left(\frac{p^\mu}{q^\nu}\right)^n \frac{q^n\phi_2(p,q) - p^{-n}\phi_1(p,q)}{q - p^{-1}} + \left(\frac{p^\mu}{q^\nu}\right)^{n+1} \frac{q^{n+1}\phi_2(p,q) - p^{-(n+1)}\phi_1(p,q)}{q - p^{-1}}. \tag{19}$$

Using (10), (19) can be re-written

$$\epsilon_n = \left(\frac{p^\mu}{q^\nu}\right)^n q^{k_0}\phi_2(p,q)\left[\left(1 + \frac{p^{\mu-1}}{q^\nu}\right)\{n - k_0\}_{p,q} + q^{(n-k_0)}\frac{p^\mu}{q^\nu}\right] \tag{20}$$

or equivalently

$$\epsilon_n = \left(\frac{p^\mu}{q^\nu}\right)^n q^{k_0}\phi_2(p,q)\left[\left(1 + \frac{p^{\mu}}{q^{\nu-1}}\right)\{n - k_0\}_{p,q} + p^{-(n-k_0)}\frac{p^\mu}{q^\nu}\right]. \tag{21}$$

It follows from (20) and (21) that the spectrum of the Hamiltonian (17) is symmetric under the change $q \to p^{-1}$, $p \to p^{-1}$, $\mu \to \nu$ and $\nu \to \mu$.

4. Generalized bibasic hypergeometric functions

On the space Γ of all finite linear combinations of the monomials z^n, $z \in \mathbb{C}$, $n \in \mathbb{Z}$:

$$\Gamma = \left\{\sum_{n\in P} a_n z^n;\, a_n \in \mathbb{C},\, P \subset \mathbb{Z}\right\}$$

we define a realization of algebras (4*a*), (4*b*) and (4*c*) in terms of the following representation:

$$A^\dagger h(z) := zh(z) \tag{22a}$$

$$Ah(z) := \frac{1}{z(p^{-1} - q)}\left(\frac{p^\mu}{q^\nu}\right)^\rho \left(p^{-\rho}\phi_1(p,q)h\left(\frac{p^{\mu-1}}{q^\nu}z\right) - q^\rho\phi_2(p,q)h\left(\frac{p^\mu}{q^{\nu-1}}z\right)\right) \tag{22b}$$

$$Nh(z) := \left(\rho + z\frac{\mathrm{d}}{\mathrm{d}z}\right)h(z). \tag{22c}$$

The action of the generators on the basis vectors of Γ, $f_m = z^n$, where $m = \rho + n$, is given by

$$A^\dagger f_m = f_{m+1} \tag{23a}$$

$$Af_m = \left(\frac{p^\mu}{q^\nu}\right)^m \frac{p^{-m}\phi_1(p,q) - q^m\phi_2(p,q)}{p^{-1} - q} f_{m-1} \tag{23b}$$

$$Nf_m = mf_m. \tag{23c}$$

It turns out that, as in the case of (p, q)-oscillator algebra, the $(p, q, \mu, \nu, \phi_1, \phi_2)$-oscillator algebra (4*a*), (4*b*) and (4*c*) reveals to be useful to construct generalized bibasic special functions. Indeed, by analogy with [5], let us define a $(p, q, \mu, \nu, \phi_1, \phi_2)$-function as follows:

$$E^{\mu,\nu}_{p,q,\phi_1,\phi_2}(z) = \sum_{n=0}^{+\infty}\left(\frac{p^\mu}{q^\nu}\right)^{n(n+1)/2}\left(\frac{q}{p}\right)^{n(n-1)/2}\frac{z^n}{[p,q;\,p,q]_n^{\phi_1,\phi_2}}$$

$$|p^\mu| < |q^{\nu-1}| \quad |pq| < 1 \quad (\mu,\nu) \neq (-1,2), \tag{24}$$

where

$$[p^\alpha, q^\beta; p, q]_n^{\phi_1,\phi_2} = \left(\frac{1}{p^\alpha}\phi_1(p,q) - q^\beta\phi_2(p,q)\right)\left(\frac{1}{p^{\alpha+1}}\phi_1(p,q) - q^{\beta+1}\phi_2(p,q)\right) \ldots \left(\frac{1}{p^{\alpha+n-1}}\phi_1(p,q) - q^{\beta+n-1}\phi_2(p,q)\right). \quad (25)$$

In terms of the q-shifted factorial $(a;q)_n = (1-a)(1-aq)\cdots(1-aq^{n-1})$, one can readily check that

$$[p^\alpha, q^\beta; p, q]_n^{\phi_1,\phi_2} = p^{-(n(n-1)/2+\alpha n)}\left(\frac{\phi_2(p,q)}{\phi_1(p,q)}p^\alpha q^\beta; pq\right)_n (\phi_1(p,q))^n. \quad (26)$$

Provided a $(p, q, \mu, \nu, \phi_1, \phi_2)$-generalization of the derivative, $D^{\mu,\nu}_{p,q,\phi_1,\phi_2}$, as follows

$$D^{\mu,\nu}_{p,q,\phi_1,\phi_2}h(z) := \frac{1}{z}\left(\phi_2(p,q)h\left(\frac{q^\nu}{p^{\mu-1}}z\right) - \phi_1(p,q)h\left(\frac{q^{\nu-1}}{p^\mu}z\right)\right) \quad (27)$$

its actions on $E^{\mu,\nu}_{p,q,\phi_1,\phi_2}(z)$ produces

$$D^{\mu,\nu}_{p,q,\phi_1,\phi_2}\left(E^{\mu,\nu}_{p,q,\phi_1,\phi_2}(z)\right) = -\frac{p}{q}E^{\mu,\nu}_{p,q,\phi_1,\phi_2}(z). \quad (28)$$

Obviously, for the restrictions $\phi_1(p,q) = \phi_2(p,q) \equiv 1$ and $\mu = \nu = 0$, one recovers the well known (p,q)-derivative introduced in [3].

As a result of such an extension, we then arrive at the definition of the generalized bibasic hypergeometric series:

$$\Phi\left[\begin{matrix}\underline{a} : \underline{c} \\ \\ \underline{b} : \underline{d}\end{matrix} ; q, p, \mu, \nu, \phi_1(p,q), \phi_2(p,q), z\right] = \sum_{l=0}^{+\infty}\frac{(\underline{a};q)_l(\underline{c};p)_l}{\left(\frac{\phi_2(p,q)}{\phi_1(p,q)}q;q\right)_l(\underline{b};q)_l(\underline{d};p)_l} \times \left(\frac{p^{\mu+\nu}}{q^\nu}\right)^{l(l+1)/2}\left[(-1)^l q^{l(l-1)/2}\right]^{1+m-n}\left[(-1)^l p^{l(l-1)/2}\right]^{s-r}\left(\frac{z}{\phi_1(p,q)}\right)^l, \quad (29)$$

where

$$\underline{a} = (a_1, \ldots, a_n) \quad \underline{c} = (c_1, \ldots, c_r)$$
$$\underline{b} = (b_1, \ldots, b_m) \quad \underline{d} = (d_1, \ldots, d_s) \quad (30)$$

$$(\underline{a};q)_l = (a_1;q)_l \ldots (a_n;q)_l. \quad (31)$$

Restricted to $\nu = \mu = 0$ and $\phi_1(p,q) = \phi_2(p,q) \equiv 1$, it exactly reproduces the well-known bibasic hypergeometric series [6]. Furthermore, the function $E^{\mu,\nu}_{p,q,\phi_1,\phi_2}(z)$ can be now expressed as

$$\Phi\left[\begin{matrix}- : 0 \\ \\ - : -\end{matrix} ; pq, p, \mu, \nu, \phi_1(p,q), \phi_2(p,q), pz\right] = E^{\mu,\nu}_{p,q,\phi_1,\phi_2}(z). \quad (32)$$

To proceed to the generalization of known special functions, one should stress the deformation functions ϕ_1 and ϕ_2 to satisfy concrete suitable relations. As a matter of convenience, from now on, we set $\phi_1(p,q) = \phi_2(p,q) = f(p,q)$ and $\lim_{(p,q)\to(1,1)} f(p,q) = 1$. Therefore, the function (24) becomes a (p, q, μ, ν, f)-exponential, namely

$$E^{\mu,\nu}_{p,q,f}(z) = \sum_{n=0}^{+\infty}\left(\frac{p^\mu}{q^\nu}\right)^{n(n+1)/2}\left(\frac{q}{p}\right)^{n(n-1)/2}\frac{\left(\frac{z}{f(p,q)}\right)^n}{[p,q;p,q]_n}, \quad (33)$$

where

$$[p^{\alpha}, q^{\beta}; p, q]_n = \left(\frac{1}{p^{\alpha}} - q^{\beta}\right)\left(\frac{1}{p^{\alpha+1}} - q^{\beta+1}\right)\cdots\left(\frac{1}{p^{\alpha+n-1}} - q^{\beta+n-1}\right). \tag{34}$$

The generalized bibasic hypergeometric series (29) becomes

$$\Phi\left[\begin{matrix}\underline{a} : \underline{c} \\ \\ \underline{b} : \underline{d}\end{matrix} ; q, p, \mu, \nu, f(p,q), z\right] = \sum_{l=0}^{+\infty} \frac{(\underline{a}; q)_l(\underline{c}; p)_l}{(q; q)_l(\underline{b}; q)_l(\underline{d}; p)_l}\left(\frac{p^{\mu+\nu}}{q^{\nu}}\right)^{l(l+1)/2}$$
$$\times\left[(-1)^l q^{l(l-1)/2}\right]^{1+m-n}\left[(-1)^l p^{l(l-1)/2}\right]^{s-r}\left(\frac{z}{f(p,q)}\right)^l. \tag{35}$$

It is noteworthy that, in the case $\mu = \nu = 0$ and $f(p,q) \equiv 1$, one recovers the (p,q)-analogue of the exponential defined by Floreanini *et al* [3]. In the limit $(p,q) \to (1,1)$, once z has been rescaled by $(p^{-1} - q)$, all these functions exactly tend to the ordinary exponential

$$\lim_{(p,q)\to(1,1)} E^{\mu,\nu}_{p,q,f}((p^{-1} - q)z) = \exp(z). \tag{36}$$

Let us then introduce the operators

$$\mathcal{U}^{(\mu,\nu,f)}(\alpha, \beta) = E^{\mu,\nu}_{p,q,f}(\alpha(p^{-1} - q)A^{\dagger})E^{\mu,\nu}_{p,q,f}(\beta(p^{-1} - q)A) \tag{37}$$

which, for $(p,q) \to (1,1)$, tend to the Lie group element $\exp(\alpha A^{\dagger})\exp(\beta A)$. Their matrix elements, in the representation space spanned by the vectors $f_m = z^n$, $m = n + \rho$, are defined by

$$\mathcal{U}^{(\mu,\nu,f)}(\alpha, \beta)z^n = \sum_{r=-\infty}^{+\infty} U^{(\mu,\nu,f)}_{r,n}(\alpha, \beta)z^r. \tag{38}$$

Using now (23*a*), (23*b*), (33), (26) and (35), one can straightforwardly establish the relevant relation

$$U^{(\mu,\nu,f)}_{r,n}(\alpha, \beta) = \beta^{n-r}\left(\frac{q}{p}\right)^{(n-r)(n-r-1)/2}\left(\frac{p^{\mu}}{q^{\nu}}\right)^{(n-r)(n+\rho+1)}$$
$$\times\mathcal{L}^{(n-r)}_{\rho+r}\left(\frac{-\alpha\beta p}{q}; p, q, \mu, \nu, f(p,q)\right), \tag{39}$$

where the generalized-Laguerre function $\mathcal{L}^{(\lambda)}_{\tilde{\nu}}(x; p, q, \mu, \nu, f(p,q))$ is given by

$$\mathcal{L}^{(\lambda)}_{\tilde{\nu}}(x; p, q, \mu, \nu, f(p,q)) = \frac{[p^{\lambda+1}, q^{\lambda+1}; p, q]_{\rho+r}}{[p, q; p, q]_{\rho+r}}$$
$$\times\Phi\left[\begin{matrix}(pq)^{-\tilde{\nu}} : 0 \\ \\ (pq)^{\lambda+1} : -\end{matrix} ; pq, p, \mu, \nu, f(p,q), (1 - pq)\left(\frac{p^{\mu}}{q^{\nu-1}}\right)^{\lambda+\tilde{\nu}+1} x\right]. \tag{40}$$

This provides an algebraic interpretation of a special class of the generalized bibasic functions of hypergeometric type. It can be used to obtain generating function. Indeed, from formulae [3]

$$\sum_{n=0}^{+\infty} \frac{[p^h, q^{\tau}; p, q]_n}{[p, q; p, q]_n} z^n = \frac{(pq^{\tau}_z; pq)_{\infty}}{(p^{1-h}z, pq)_{\infty}} \tag{41}$$

and [7]

$$(a; pq)_{\lambda} = \frac{(a; pq)_{\infty}}{(a(pq)^{\lambda}; pq)_{\infty}} \tag{42}$$

one shows that

$$\mathcal{U}^{(\mu,\nu,f)}(\alpha,\beta)z^n = E^{\mu,\nu}_{p,q,f}(\alpha(p^{-1}-q)z)z^n\left(-\frac{p^2\beta}{z}\left(\frac{p^{\mu-1}}{q^\nu}\right)^{n+\rho+1};pq\right)_{n+\rho}. \tag{43}$$

Inserting this result and expression (39) of the matrix elements $U^{(\mu,\nu,f)}_{r,n}(\alpha,\beta)$ in (38), and setting $n=0$, $\beta=-q/p$ and $t=-1/z$, one can deduce the searched relation

$$E^{\mu,\nu}_{p,q,f}\left(-\alpha(p^{-1}-q)/t\right)\left(-pqt\left(\frac{p^{\mu-1}}{q^\nu}\right)^{\rho+1};pq\right)_\rho = \sum_{r=-\infty}^{+\infty}\left(\frac{q}{p}\right)^{r(r+1)/2}\left(\frac{p^\mu}{q^\nu}\right)^{r(\rho+1)} \times \mathcal{L}^{(r)}_{\rho-r}(\alpha;p,q,\mu,\nu,f(p,q))\,t^r. \tag{44}$$

For $(p,q)\to(1,1)$, this equation reduces to the generating relation [8]

$$\mathrm{e}^{-\alpha/t}(1+t)^\rho = \sum_{r=-\infty}^{+\infty} t^r L^{(r)}_{\rho-r}(\alpha) \tag{45}$$

for the usual Laguerre functions $L^{(m)}_\lambda$. Besides, taking $\rho=0$ and restricting to analytic functions, one obtains, from the above-generalized algebra, a representation bounded below. Using $f_m := z^m$, with $m\in\mathbb{Z}^+$ as basis vectors, the matrix elements of $\mathcal{U}^{(\mu,\nu,f)}(\alpha,\beta)$ now defined as

$$\mathcal{U}^{(\mu,\nu,f)}(\alpha,\beta)z^n = \sum_{r=0}^{+\infty} U^{(\mu,\nu,f)}_{r,n}(\alpha,\beta)z^r \tag{46}$$

are simply obtained by setting $\rho=0$ in (39). Since for $\tilde{\nu}$ integer, $\mathcal{L}^{(\lambda)}_{\tilde{\nu}}(x;p,q,\mu,\nu,f)$ is a polynomial of order $\tilde{\nu}$, the matrix elements $U^{(\mu,\nu,f)}_{r,n}(\alpha,\beta)$ are here expressed in terms of generalized-Laguerre polynomials, called (p,q,μ,ν,f)-Laguerre polynomials. A generating function for these polynomials can be obtained as follows. If $\rho=0$, by substituting β by q/p and α by $-\alpha$, we obtain

$$E^{\mu,\nu}_{p,q,f}(-\alpha(p^{-1}-q)z)z^n\left(-\frac{pq}{z}\left(\frac{p^{\mu-1}}{q^\nu}\right)^{n+1};pq\right)_n = \sum_{r=0}^{+\infty}\left(\frac{q}{p}\right)^{(n-r)(n-r+1)/2} \times\left(\frac{p^\mu}{q^\nu}\right)^{(n-r)(n+1)}\mathcal{L}^{(n-r)}_r(\alpha;p,q,\mu,\nu,f(p,q))\,z^r, \tag{47}$$

which is nothing but a (p,q,μ,ν,f)-analogue of relation [8]

$$e^{-\alpha z}(1+z)^n = \sum_{r=0}^{+\infty} L^{(n-r)}_r(\alpha)z^r \tag{48}$$

for ordinary Laguerre polynomials $L^{(m)}_\lambda$ to which (47) reduces when $(p,q)\to(1,1)$.

To conclude, let us point out that at first sight the function $f(p,q)$ in the generalized hypergeometric series (35) can be eliminated by a rescaling of the variable z/f by writing, for instance, $\tilde{z}=z/f$. But, a closer look on the development performed in the following shows that one immediately leads to the appearance of the same function f in the arguments of the generalized Laguerre functions (39)–(40) as well as in the associated subsequent relations. Hence, a choice has to be made. One can also imagine that the generalized exponentials (33) can be subjected to the same rescaling, what does not correspond to its definition which follows from the representation (22)–(23) of the generalized oscillator algebra (4*a*)–(4*c*) in the space of monomials of complex variables when $\phi_1=\phi_2=f$.

Finally, it would be of some interest to investigate the classes of operators which generate such a $(p, q, \mu, \nu, \phi_1, \phi_2)$-generalized oscillator algebra and the conditions of its closure. In the same vein, the search for specific relations between ϕ_1 and ϕ_2 to generate generalizations of all known special functions could be envisaged. These questions are now under consideration.

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References

[1] Chakrabarti R and Jagannathan R 1991 A (p, q) oscillator. Realisation of two-parameter quantum algebras *J. Phys. A: Math. Gen.* **26** L711–8

[2] Arik D D, Coon Y and Lam A 1975 Operatr algebra of dual resonance *J. Math. Phys.* V **16** 1776

[3] Floreanini R, Lapointe L and Vinet L 1993 A note on (p, q)-oscillators and bibasic hypergeometric functions *J. Phys. A: Math. Gen.* **26** 611–4

[4] Naderi M H, Soltanolkotabi M and Roknizadeh R 2004 On a family of photon-added deformed coherent states associated with two-parameter deformed boson oscillator algebra *Can. J. Phys.* **82** 632–46

[5] Floreanini R, LeTourneux and Vinet L 1995 More on the q-oscillator algebra and q-orthogonal polynomials *J. Phys. A: Math. Gen.* **28** L287–93

[6] Gasper G and Rahman M 1990 *Basic Hypergeometric Series* (Cambridge: Cambridge University Press)

[7] Koekoek R and Swarttouw R F 1994 *The Askey Scheme of Hypergeometric Orthogonal Polynomials and the q-analogue Report 94-05* Delft University of Technology

[8] Erdelyi A 1953 *Higher Transcendental Functions* (New York: McGraw-Hill)

[9] Moak D S 1981 The q-analogue of the Laguerre polynomials *J. Math. Anal. Appl.* **81** 20–47

[10] Biedenharn L C 1989 The quantum group $SU_q(2)$ and a q-analogue of the boson operators *J. Phys. A: Math. Gen.* **22** L873

[11] Macfarlane A J 1989 On q-analogues of the quantum harmonic oscillator and quantum group $SU(2)_q$ *J. Phys. A: Math. Gen.* **22** 4581

[12] Floreanini R and Vinet L 1994 $\mathcal{U}_q(sl(2))$ and q-special functions (Lie algebras, cohomology and new applications to quantum mechanics) *Contemp. Math.* **160** 85–160